College of Higher Vocational Studies

Exercise 9

(1) Consider the Heaviside function

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x \le 0, \end{cases}$$

and let δ be the Dirac measure $\delta\varphi = \varphi(0)$, $\forall \varphi \in \mathcal{D}$.

(a) Show that
$$(H * \varphi)(x) = \int_{-\infty}^{x} \varphi(s)\, d\mathcal{L}^1(s), \quad \forall \varphi \in \mathcal{D}(\mathbb{R})$$

(b) Show that
$$\delta' * H = \delta.$$

(c) Show that $1 * \delta' = 0$ (1 is the constant function)

(2) Let $f \in L^1(\mathbb{R}^n)$. Show that

(a) $\lim_{x \to 0} \|\tau_x f - f\|_{L^1} = 0$

(b) Deduce from (a) that $\hat{f} \in C(\mathbb{R}^n)$.

(c) Show that $\hat{f} \in C_0(\mathbb{R}^n)$. ($g \in C_0(\mathbb{R}^n)$ means $g \in C(\mathbb{R}^n)$ and $\forall \varepsilon > 0$, $\exists$ cpt K s.t. $|g(x)| < \varepsilon$, $\forall x \notin K$.)

(3) Show that $\mathcal{S}$ is a Fréchet space. A sequence $\{f_k\}$ in $\mathcal{S}$ is a Cauchy sequence if $\forall (m, N)$, $\forall \varepsilon > 0$, $\exists n_0$ s.t.
$$\|f_k - f_\ell\|_{m,N} < \varepsilon, \quad \forall k, \ell \ge n_0.$$

Tel: 2788 8524 Fax: 2788 9315 Web Site: http://www.cityu/edu.hk/col

where

$$\|f\|_{m,N} = \sup_x (1+|x|^2)^{\frac{m}{2}} \sum_{|\alpha|\le N} |D^\alpha f(x)| .$$

$\mathcal{S}$ is Fréchet means that every Cauchy sequence converges, that's,

$\exists f \in \mathcal{S}$ s.t. $\|f_k - f\|_{m,N} \to 0$ as $k \to \infty$ for every (m, N).

(4) Prove Proposition 7 (a), that is,

$$\phi * h_j \longrightarrow \phi \text{ in } \mathcal{D}, \quad \forall \varphi \in \mathcal{D}.$$

for any approximate identity $\{h_j\}$.

(5) Provide a proof for Proposition 14.

Tel: 2788 8524 Fax: 2788 9315 Web Site: http://www.cityu/edu.hk/col